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LETTER TO THE EDITOR

Canonical transformation and the topological term in one-dimensional antiferromagnets

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Abstract. A canonical transformation is found which generates the topological term in a one-dimensional antiferromagnetic chain.

Much progress has been made in studying the one-dimensional Heisenberg antiferromagent [1-4]. The model was found to be the O(3) non-linear σ model by Haldane [1]. The continuum dynamics of the model was also established for the quantum version in [2]. Affleck gave the topological term in the continuous limit [3]. Recently, the topological term was also derived by Fradkin and Stone [4] on the basis of the rotation operators and the path integral. In this letter we shall show that the topological term in the one-dimensional Heisenberg antiferromagnetic chain originates in a kind of canonical transformation within the framework of Haldane's approach [1, 2].

For simplicity we consider the Hamiltonian

$$H=\sum_n \boldsymbol{S}_n\cdot\boldsymbol{S}_{n+1} \tag{1}$$

where in the classical limit

$$\boldsymbol{S}_n=(-1)^n S(\sin\theta_n\cos\varphi_n, \sin\theta_n\sin\varphi_n, \cos\theta_n) \tag{2}$$

and

$$\{S_n^a, S_m^b\}=\varepsilon^{abc}S_n^c\delta_{mn}. \tag{3}$$

The equation of motion [1] is

$$\dot\theta_n=-S(-1)^n\sum_n \sin\theta_{n\pm1}\sin(\varphi_{n\pm1}-\varphi_n) \tag{4}$$

$$\dot\varphi_n=-S(-1)^n\sum_n[\cos\theta_{n\pm1}-\cot\theta_n\sin\theta_{n\pm1}\cos(\varphi_{n\pm1}-\varphi_n)]. \tag{5}$$

Considering the fluctuation of the field [1]

$$\theta_n=\theta(x)+a(-1)^n\alpha(x) \qquad \varphi_n=\varphi(x)+a(-1)^n\beta(x) \tag{6}$$

with a being the lattice spacing, and introducing the canonical momenta

$$\pi_\theta=S_\beta\sin\theta \qquad \pi_\varphi=-S^\alpha\sin\theta \tag{7}$$

which satisfy the Poisson brackets

$$\begin{gathered}\{\theta(x), \varphi(x')\}=\{\pi_{\theta(x)}, \pi_{\varphi(x')}\}=\{\varphi(x), \pi_{\theta(x')}\}=\{\theta(x), \pi_{\varphi(x')}\}=0\\ \{\theta(x), \pi_{\theta(x')}\}=-\{\varphi(x), \pi_{\varphi(x')}\}=\delta(x-x')\end{gathered} \tag{8}$$

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the Hamiltonian can be recast in the form

$$H=\int \mathrm{d}x\{(gc/2)(\pi_\theta^2+\pi_\varphi^2/\sin^2\theta)+(c/2g)[(\partial\theta)^2+\sin^2\theta(\partial\varphi)^2]\} \tag{9}$$

with

$$c=2aS \qquad g=2/S \qquad \partial=\partial_x.$$

The corresponding Lagrangian is the same as that of the usual O(3) non-linear σ model

$$\mathscr{L}=(c/2g)(\partial_\mu\mathbf{\Omega})^2 \tag{10}$$

where

$$\mathbf{\Omega}=\{(1-\Omega^2)^{1/2}\cos\varphi,(1-\Omega^2)^{1/2}\sin\varphi,\Omega\} \qquad \cos\theta=\Omega. \tag{11}$$

The key point in constructing the above Hamiltonian system is based on the mapping (2) which maps the base spacetime onto the spin sphere with

$$\theta_n=\theta_n(x,t) \qquad \varphi_n=\varphi_n(x,t). \tag{12}$$

This allows the existence of a new type of canonical transformation for the fibre bundle

$$\begin{aligned}&\varphi\to\varphi \qquad \theta\to\theta\\ &\pi_\varphi\to\tilde{\pi}_\varphi=\pi_\varphi+k\partial\Omega\\ &\pi_\Omega\to\tilde{\pi}_\Omega=\pi_\Omega-k\partial\varphi.\end{aligned} \tag{13}$$

It is easy to verify that the transformation (13) preserves the Poisson bracket, i.e. it is a canonical transformation. Actually we have

$$\begin{aligned}&\{\varphi(x),\tilde{\pi}_{\varphi(x')}\}=\{\Omega(x),\tilde{\pi}_\Omega(x')\}=\delta(x-x')\\ &\{\tilde{\pi}_\varphi(x),\tilde{\pi}_\Omega(x')\}=k\partial\delta(x-x')+k\partial'\delta(x-x')=0\end{aligned} \tag{14}$$

where (8) has been used. In accordance with the equation of motion

$$\dot{\varphi}=\{\varphi,\tilde{H}\} \qquad \dot{\Omega}=\{\Omega,\tilde{H}\} \qquad \dot{\tilde{\pi}}_\varphi=\{\tilde{\pi}_\varphi,\tilde{H}\} \qquad \dot{\tilde{\pi}}_\Omega=\{\tilde{\pi}_\Omega,\tilde{H}\} \tag{15}$$

the transformed Hamiltonian has the form

$$\tilde{H}=\int \mathrm{d}x\,\tilde{\mathscr{H}}=\int \mathrm{d}x\left[\frac{gc}{2}(1-\Omega^2)[\tilde{\pi}_\Omega+k\partial\varphi]^2+\left(\frac{(\tilde{\pi}_\varphi-k\partial\Omega)^2}{1-\Omega^2}\right)+\frac{c}{2g}(\partial\mathbf{\Omega})^2\right]. \tag{16}$$

It follows that

$$\tilde{H}=H=\int \mathrm{d}x\,\mathscr{H}. \tag{16'}$$

The corresponding Lagrangian in this case is

$$\tilde{\mathscr{L}}=\dot{\Omega}\tilde{\pi}_\Omega+\dot{\varphi}\tilde{\pi}-\tilde{\mathscr{H}} \tag{17}$$

which leads to

$$\tilde{\mathscr{L}}=\frac{c}{2g}(\partial_\mu\mathbf{\Omega})^2+\frac{kc}{2}\varepsilon^{\mu\nu}\mathbf{\Omega}\cdot(\partial_\mu\mathbf{\Omega}\times\partial_\nu\mathbf{\Omega}) \qquad \mu,\nu=0,1 \tag{18}$$

where in deriving (18) we have used the equality

$$\frac{c}{2}\varepsilon^{\mu\nu}\mathbf{\Omega}\cdot(\partial_\mu\mathbf{\Omega}\times\partial_\nu\mathbf{\Omega})=\sin\theta(\dot{\theta}\partial\varphi-\dot{\varphi}\partial\theta). \tag{19}$$

The meaning of the canonical transformation (13) is clear from the point of view of analytic mechanics that relating to a canonical transformation there is a generating function f such that

$$\mathrm{d}f=(\tilde{\pi}_\varphi-\pi_\varphi)\,\mathrm{d}\varphi+(\tilde{\pi}_\Omega-\pi_\Omega)\,\mathrm{d}\Omega \tag{20}$$

which is valid for a Hamiltonian system dependent on t through θ and φ only. Now the system is canonical with respect to θ and φ, regarding x as a parameter; hence we have

$$\mathrm{d}\varphi=\dot{\varphi}\,\mathrm{d}t \qquad \mathrm{d}\Omega=\dot{\Omega}\,\mathrm{d}t. \tag{21}$$

By using (19) we obtain

$$f=(-kc/2)\int \varepsilon^{\mu\nu}\boldsymbol{\Omega}\cdot(\partial_\mu\boldsymbol{\Omega}\times\partial_\nu\boldsymbol{\Omega})\,\mathrm{d}t \tag{22}$$

i.e., the generating function has the form

$$F=\int f\,\mathrm{d}x=(-kc/2)\int\int \mathrm{d}x\,\mathrm{d}t\,\varepsilon^{\mu\nu}\boldsymbol{\Omega}\cdot(\partial_\mu\boldsymbol{\Omega}\times\partial_\nu\boldsymbol{\Omega}) \tag{23}$$

which after Wick rotation gives exactly the topological term. The result is easy to understand because the generating function receives the boundary contribution and the transformation (13) is related to a spin sphere.

For the quantum version we have [2]

$$[S_n^a, S_m^b]=\mathrm{i}\varepsilon^{abc}S_n^c\delta_{nm} \qquad S_n^2=S(S+1)$$
$$[\varphi_n, S_m^Z]=\mathrm{i}\delta_{nm}. \tag{24}$$

Haldane introduced the Fourier decompositions in [2]

$$\varphi_n=n\pi+N_a^{-1/2}\sum_q\exp(\mathrm{i}qna)\phi_q \qquad S_n^z=N_a^{-1/2}\sum_q\exp(-\mathrm{i}qna)L_q \tag{25}$$

where N_a is the number of sites and a is the lattice spacing. On a length scale [2]

$$\begin{aligned}\varphi_n&=n\pi+\varphi(x)-a(-1)^n[S(S+1)]^{-1/2}\pi(x)\\ S_n^Z&=aL(x)+(-1)^n[S(S+1)^{1/2}\Omega(x)\end{aligned} \tag{26}$$

with

$$[\varphi(x,t), L(x',t)]=[\Omega(x),\pi(x')]=\mathrm{i}\delta(x-x'). \tag{27}$$

By analogy with the classical case, it allows the existence of the transformation

$$\pi\to\tilde{\pi}=\pi+k\partial\varphi \qquad L\to\tilde{L}=L-k\partial\Omega \tag{28}$$

which preserves the canonical commutators. In this way we rederive the Lagrangian (18). Correspondingly under (28), equations (26) and (27) become

$$\begin{aligned}\tilde{\varphi}_n&=n\pi+\varphi(x-ak(-1)^n/\sqrt{S(S+1)})-a(-1)^n[S(S+1)]^{-1/2}\pi(x)\\ \tilde{S}_n^Z&=aL(x)+(-1)^n[S(S+1)]^{1/2}\Omega(x-ak(-1)^n/\sqrt{S(S+1)}).\end{aligned} \tag{29}$$

It is seen by comparing (29) and (26) that the canonical transformation equivalently forces the continuous limit to take not the substitution $x=na$ for $\varphi(x)$ and $\Omega(x)$ but rather

$$\begin{aligned}\varphi(x)&\to\varphi(x-ak(-1)^n/\sqrt{S(S+1)})\\ \Omega(x)&\to\Omega(x-ak(-1)^n/\sqrt{S(S+1)})\end{aligned} \tag{30}$$

for the antiferromagnetic chain. Our results coincide with those of Affleck [3] by

$$k = 1/g \qquad \text{or} \qquad k/\sqrt{S(S+1)} = \tfrac{1}{2}. \tag{31}$$

In the language of physics, the canonical transformation makes the opposite spins at the neighbouring sites tend toward a pair at the middle point of the spacing. Thus we can think of the topological term as relating to the fact that the neighbouring spins on the lattices pair up for an antiferromegnetic chain. We believe that the approach discussed here can be extended to other models, for example for the lattice fermion operators which leads to the continuum operators $\psi_R(x)$, $\psi_L(x)$ in the following way [4, 5]:

$$\psi_{R,n} = a^{1/2}\psi_R(x+\xi a) \qquad \psi_{L,n} = a^{1/2}\psi_L(x-\xi a) \tag{32}$$

where $x = na$ and $0 < \xi < \frac{1}{2}$, as shown by de Vega [5].

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